

What's another way to make 18?

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Activity:

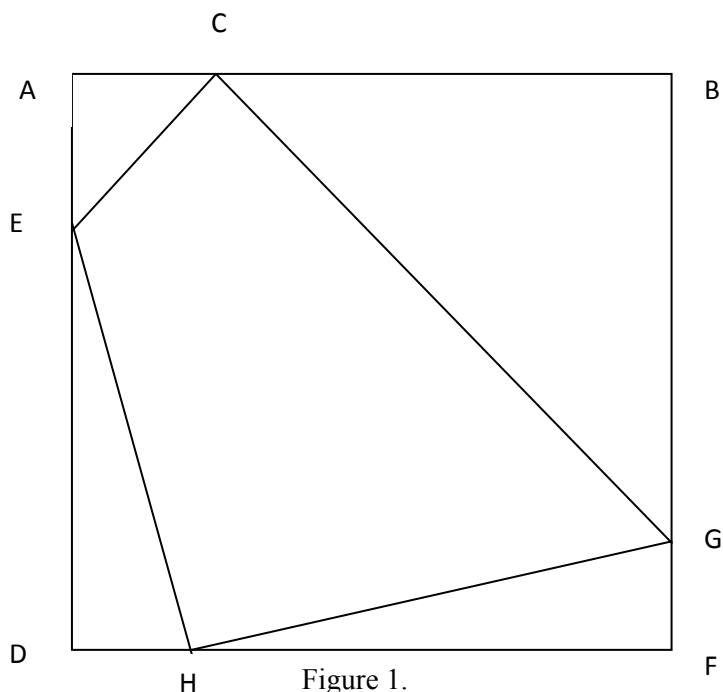
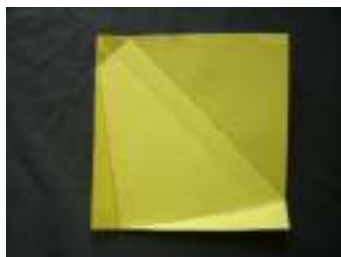
Take a 6 centimetre by 6 centimetre piece of paper, and after following the instructions on folding that piece of paper, compare yours with your neighbours or make another with a different length for the first fold.

Instructions

Create a fold so that $AC = AE$.

Then a fold such that $BC = BG$.

Finally, fold so that $DH = FG$.



The question now asked is, “Can you suggest something about the areas of the quadrilaterals that have been formed inside the original square?”

One suggestion is that they may be equal. We will call it ‘The Paper-folding Conjecture’ and investigate this by using the geometrical animation capability of the ClassPad330+, to show a series of differently folded squares to produce quadrilaterals, with two examples shown in Figure 1.

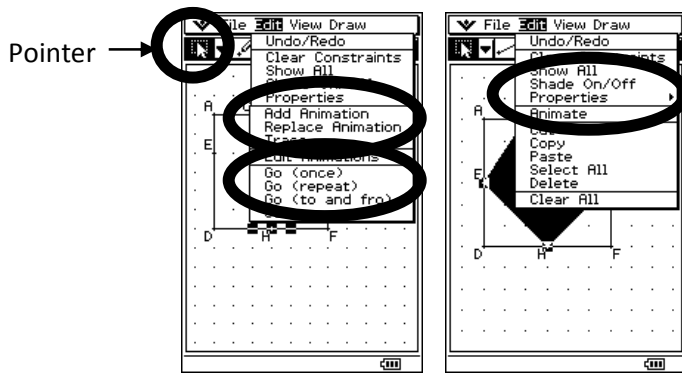
The construction sequence

Discuss how you could approach the setting up of this animation on the ClassPad330+. Where do the position of point E, F, G and H start in the animation?

A series of instruction sequences follow, beginning with drawing the line AB and then place C anywhere on the line AB, as the animation always begins from the starting point from which the line was first constructed.

- Draw the line AB and draw a point C

- Change the tool to a 'pointer'. See Figure 2a
- Tap a point C, then the line AB
- Open **Edit** and selecting **Add animation**. See Figure 2b
- Repeat this process for producing the line AD and the point E and add the animation.
- Create the line FB and the point G and add the animation
- Construct the line DF and point H and add the animation
- Then form the quadrilateral 'snapping the vertices' to the points EFHG. Add some shading to distinguish the area of the quadrilateral by selecting the four vertices [changing the tool to 'pointer'] and opening **Edit** then selecting **Shade On/Off**. See Figure 2b.



a Figure 2. b

Below, in Figure 3, are some screen captures illustrating parts of the construction sequence and the Drop-down tab to access the construction tools for line segments and points.

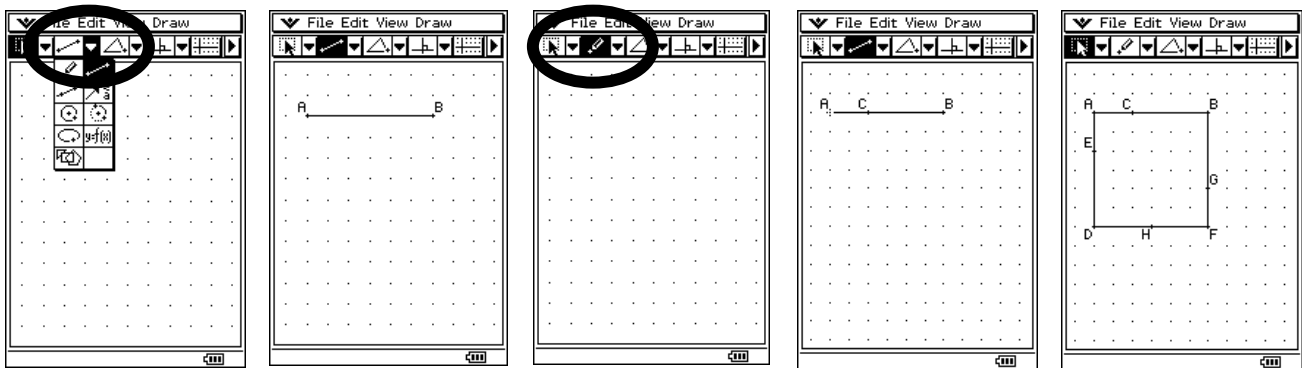


Figure 3. Construction of the line AB and a point C and the completed square.

Select from the 'Tool Bar' by tapping on the drop down tab to the right and select the 'line segment' tool. In the working area, tap once and this will produce point A, then tap again 6 units from point A, this will produce the line AB. From the 'Tool Bar' again select the 'point' icon. Repeat the use of the 'finite line' and 'point' icons to create the square and the points that form the quadrilateral, as shown in the last frame of Figure 3. The construction of the square is important for the animation of the points. The example shown has the sides constructed in the order AB, AD, DF and FG.

The Animation captured

When the construction and animation of the square, points and the quadrilateral are completed, the animation can be started by opening **Edit** and selecting **Animation** then **Go (once)**.

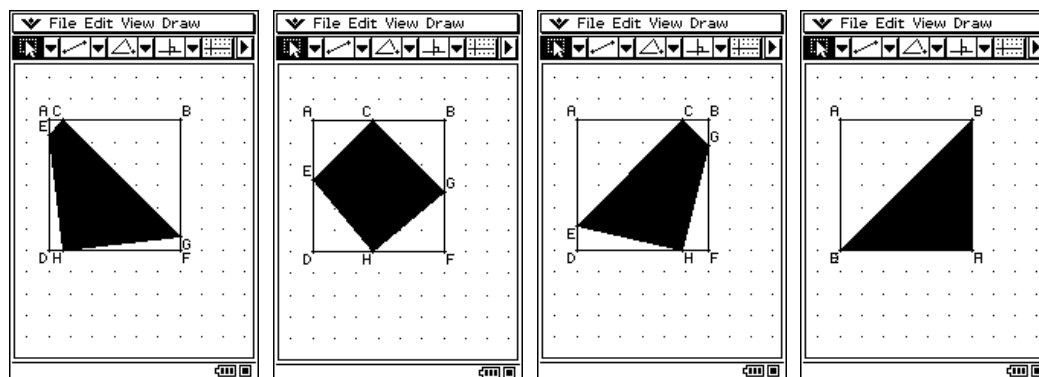


Figure 4. Screen captures of the animation sequence.

Collecting Table values from the changing quadrilateral

Running the animation creates a 'log of data values' that have been collected from the animation process, these include angles, lengths, gradients and areas. Now, tap on the arrow on the far left of the 'Tool Bar' to view another 'page', see figure 5.

Select the points C, E, H and G by tapping on these points to highlight. The number 18 will be displayed, as well as an 'area' icon to the left. To generate the table of the changing quadrilaterals area, tap on the '2 column' icon, to the right of 18, as indicated in Figure 5.

This has confirmed 'The Paper-folding Conjecture', that the area was indeed $\frac{1}{2}$ of 36 [18] by examples.

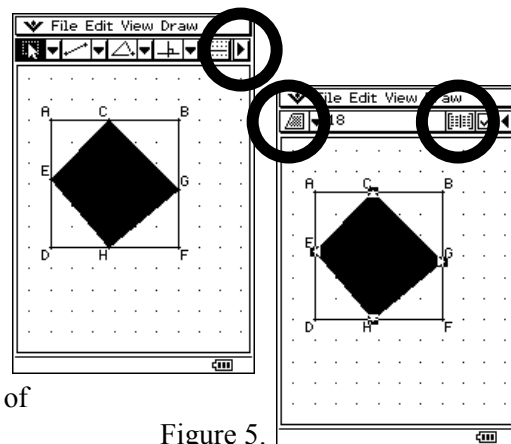


Figure 5.

Explore what is happening to the four triangular areas that have been formed inside the original square by selecting the triangles in turn and creating a table of values for these respective areas as illustrated in Figure 6.

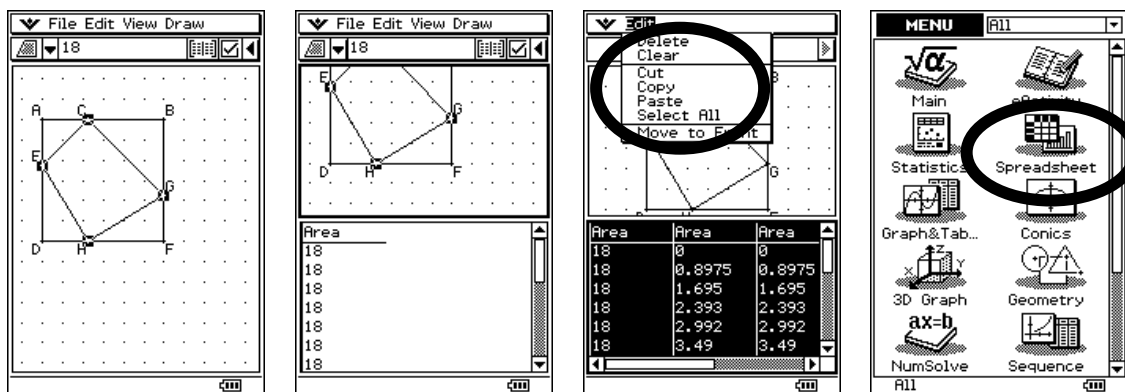


Figure 6. Collecting the data from the animations of the areas.

All of the data collected from the animations [Select All, Copy and Paste via the Edit dropdown] can be placed in the Spreadsheet and confirmed, to 'limits of accuracy' from the data that the five areas collectively [four triangles and quadrilateral] added to 36 cm^2 .

The algebra

Question: "What is another way to write 18?"

Working: Original area = 36

Let the length AC be x .

$$\text{Area of triangle AEC} = \frac{1}{2} x^2$$

$$\text{Area of triangle CBG} = \frac{1}{2} (6 - x)(6 - x)$$

$$\text{Area of triangle GFH} = \frac{1}{2} (6 - x) x$$

$$\text{Area of triangle HDE} = \frac{1}{2} (6 - x) x$$

Based on the animation, the area of quadrilateral CGHE is possibly 18, as the table capture in figure 6 illustrates.

$$\text{Area of quadrilateral} = \text{Area } \square\text{ABFD} - \text{area } \triangle\text{AEC} - \text{area } \triangle\text{CBG} - \text{area } \triangle\text{GFH} - \text{area } \triangle\text{HDE}$$

Affirming the algebra

$$\begin{aligned} \text{Area of quadrilateral} &= 36 - \frac{1}{2}x^2 - \frac{1}{2}(6-x)(6-x) - \frac{1}{2}(6-x)x - \frac{1}{2}(6-x)x \\ &= 36 - \frac{1}{2}x^2 - \frac{1}{2}(36 - 12x + x^2) - \frac{1}{2}(6x - x^2) - \frac{1}{2}(6x - x^2) \\ &= 36 - \frac{1}{2}x^2 - 18 + 6x - \frac{1}{2}x^2 - 3x + \frac{1}{2}x^2 - 3x + \frac{1}{2}x^2 \\ &= 36 - 18 - \frac{1}{2}x^2 - \frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1}{2}x^2 + 6x - 3x - 3x \\ &= 18 \text{ cm}^2 \end{aligned}$$

Using the **Main** icon, confirmed the algebra illustrated above. Using the ‘expanding’ and ‘simplifying’ actions from the Drop-down tabs, as illustrated in Figure 7. The expression for the relationship found between the original square, the quadrilateral and the four triangles can be confirmed again.

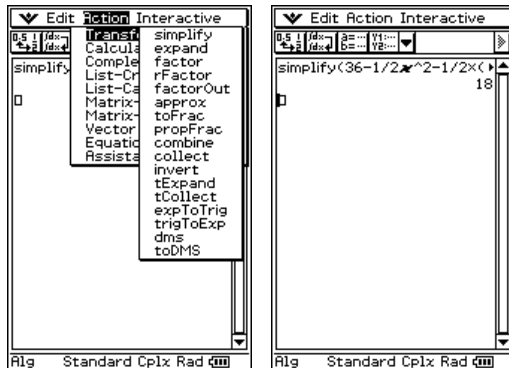


Figure 7.

This activity links the concepts of geometrical construction with animation and the logic and reasoning required the construction sequence. The animation demonstrated a number of possible constructions based on a varying lengths of the original CE fold. The use of area formulae and calculations along with algebra confirmed that the area of the quadrilateral created was $\frac{1}{2}$ of the squares original area.

Extension ideas

1. Would this construction work for any dimension of the original square? Can it be generalised?
2. What would be the result if the original paper piece was rectangular?