Nth roots of a Complex Number - $z^{n} = (a + \boldsymbol{i}\, b)$

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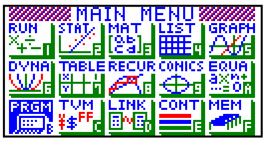
Select PRGM icon (press B) from the main menu or by using the arrow keys to highlight and then press EXE.



De Moive's Theorem will be very helpful here.

$z^{n} = (a + \boldsymbol{i}\, b)^{n} = (r\cos\theta + \boldsymbol{i}\sin\theta)^{n} = r^{n}(\cos n\theta + \boldsymbol{i}\sin n\theta)$

So to find the nth root of a complex number we get:

$z = (z^{n})^{1/n} = ((a + \boldsymbol{i}\, b)^{n})^{1/n}$

$\quad = ((r\cos\theta + \boldsymbol{i}\sin\theta)^{m})^{1/n}$

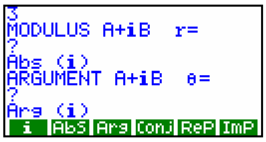
$\quad = r^{m/n}(\cos({}^{m\theta}/_{n} + {}^{2m\pi}/_{n}) + \boldsymbol{i}\sin({}^{m\theta}/_{n} + {}^{2m\pi}/_{n}))$

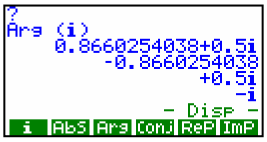
This can be written in shortened notation (polar form): $z = r^{m/n}\mathrm{cis}({}^{m\theta}/_{n} + {}^{2m\pi}/_{n})$

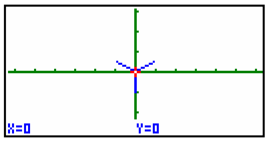
For example:

Solve $z^{3} = \boldsymbol{i}$, as $\boldsymbol{i} = 0 + 1\,\boldsymbol{i}$, the modulus of z^{3} is $r = \sqrt{(0^{2} + 1^{2})} = 1$ and the argument is ${}^{\pi}/_{2}$, as $\tan(\theta) = {}^{1}/_{0}$, implying $\theta = {}^{\pi}/_{2}$.

Yielding $z = \mathrm{cis}({}^{\pi}/_{6} + {}^{2m\pi}/_{3})$, m = 0, 1, 2, 3, … **BUT** m = 0, 1, 2 will reveal the 3 roots to $z^{3} = \boldsymbol{i}$, as when m = 3, this will be the same answer to m = 0, similarly m = 4 gives the same answer when m = 1 and so on.







Solutions:			
	$z_0 = 0.866 + 0.5\,\boldsymbol{i}$	$z_1 = -0.866 + 0.5\,\boldsymbol{i}$	$z_2 = -\,\boldsymbol{i}$
	$[z_0 = {}^{\sqrt{3}}/_{2} + 0.5\,\boldsymbol{i}$	$z_1 = -\,{}^{\sqrt{3}}/_{2} + 0.5\,\boldsymbol{i}$	$z_2 = -\,\boldsymbol{i}]$
	$[z_0 = \mathrm{cis}({}^{\pi}/_{6})$	$z_1 = \mathrm{cis}({}^{5\pi}/_{6})$	$z_2 = \mathrm{cis}({}^{9\pi}/_{6})]$

Write the following program as a new file in **PRGM**.

Filename:COMPLEX

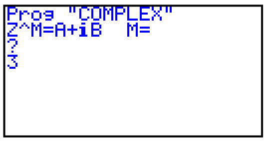
```
Deg
0→A~Z
"Z^M=A+B  M ="
? → M
"MODULUS A+,B  r ="
? → r
"ARGUMENT A+,B  θ ="
? → θ
ViewWindow -6.3,6.3,1,-3.1,3.1,1
Lbl 1
1÷Mx( θ +2xNx180) → A
r^(1÷M)(cos A+,sin A)◄
Plot 0,0
Plot r^(1÷M)cos A, r^(1÷M)sin A
Line·
N+1 → N
N=M+1=>Goto 2
Goto 1
Lbl 2
Plot 0,0
```

Example:

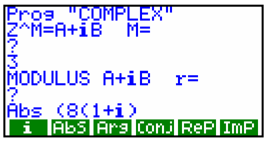
Solve $z^3 = 8(1 + \boldsymbol{i})$

Run the program called **"COMPLEX"** via MAIN MENU ICON **PRGM**

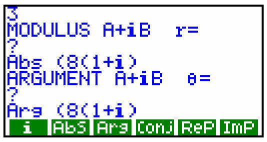
Enter in 3 **EXE**
This is the power.



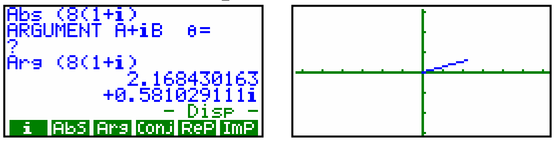
Enter in the modulus [Abs]
Using **OPTN** **F3** CPLX **F2**



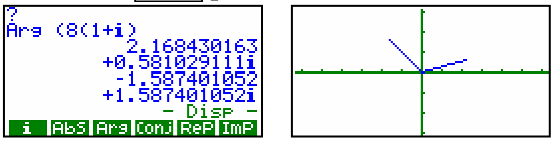
Enter in the argument [Arg]
Using **OPTN** **F3** CPLX **F3**



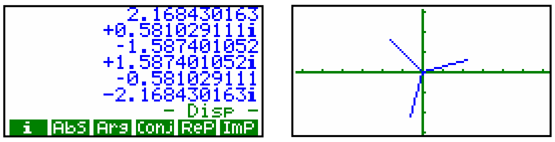
The 'screensnaps' that follow are shown below, after each **EXE** press:



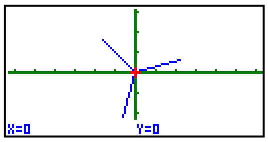
$z_0 = 2.168 + 0.581\ \boldsymbol{i}$



$z_1 = -1.587 + 1.587\ \boldsymbol{i}$



$z_2 = -0.581 + -2.168\ \boldsymbol{i}$



The 3 roots of $8(1 + \boldsymbol{i})$

Points to note: all the roots are symmetrically placed around the origin (0,0), i.e. the angle between each of the roots is 120º, all have the same modulus i.e. 2√2

Try solving

1. $z^4 = 1$
2. $z^3 = \boldsymbol{i}$
3. $z^6 = -1$
4. $z^4 = 16(1 - \boldsymbol{i})$

Answers: