

## Moessner's Theorem.

*This resource was written by Derek Smith with the support of CASIO New Zealand. It may be freely distributed but remains the intellectual property of the author and CASIO.*

Moessner's Theorem describes a procedure for generating a  $n$  integer sequences that lead unexpectedly to the sequence of  $n$ th powers  $1^n, 2^n, 3^n, \dots$  for  $n > 1$ .  
 To generate the first sequence, write down the positive integers  $1, 2, 3, \dots$ , then cross out every  $n^{\text{th}}$  element. For the second sequence, compute the prefix sums of the first sequence, ignoring the crossed-out elements, then cross out every  $(n - 1)^{\text{th}}$  element. For the third sequence, compute the prefix sums of the second sequence, then cross out every  $(n - 2)^{\text{th}}$  element, and so on until you have only one number for each 'n-group' that you started with.

For example, for  $n = 4$ :

1	2	3	4	5	6	7	8	9	10	11	<del>12</del>	13	14	15	<del>16</del>	17	18	19	<del>20</del>	21	22	23	24
1	3	6		11	17	24		33	43	54		67	81	96		113	131	150		171	193	216	
1	4		15	32		65	108		175	256		369	500		671	864							
1		16		81		256		625		1296													

Moessner's theorem says that the final sequence is  $1^n, 2^n, 3^n, \dots$

**Checking:**  $1 = 1^4, 16 = 2^4, 81 = 3^4, 256 = 4^4, 625 = 5^4, 1296 = 6^4$



$1^4$  1	$2^4$  16	$3^4$  81	$4^4$  256
----------------	-----------------	-----------------	------------------

**Try for  $n = 3$**

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

**Try for  $n = 5$**

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

**Try for  $n = 6$**

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
---	---	---	---	---	---	---	---	---	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

**Questions:**

1. What happens when  $n = 1$ ?
2. What is the value of  $n$  so that the 3<sup>rd</sup> and 4<sup>th</sup> number in the last sequence generated are 729 and 6 561 respectively?
3. In the construction of Moessner's Theorem, the initial step size  $n$  is constant.
  - (a) What would happen if you increase it at each step? Repeat the construction starting with a step size of one and increasing the step size by one each time. Thus, in the first sequence, we cross out 1, 3, 6, 10, ... What sequence of numbers are generated now?
  - (b) What would happen if you increased  $n$  as an arithmetic sequence? Thus, in the first sequence, we cross out 1, 3, 5, 7, ... What sequence of numbers are generated now?

Hint: Factorials.  $3! = 3 \times 2 \times 1 = 6$

Hint:  $a \times 1^{n-1}, (a + d) \times 2^{n-1}, (a + 2d) \times 3^{n-1}, \dots$