Turning triangles into a square!

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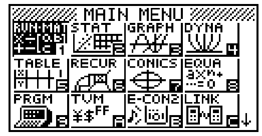
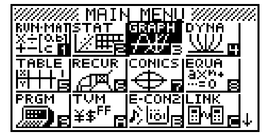
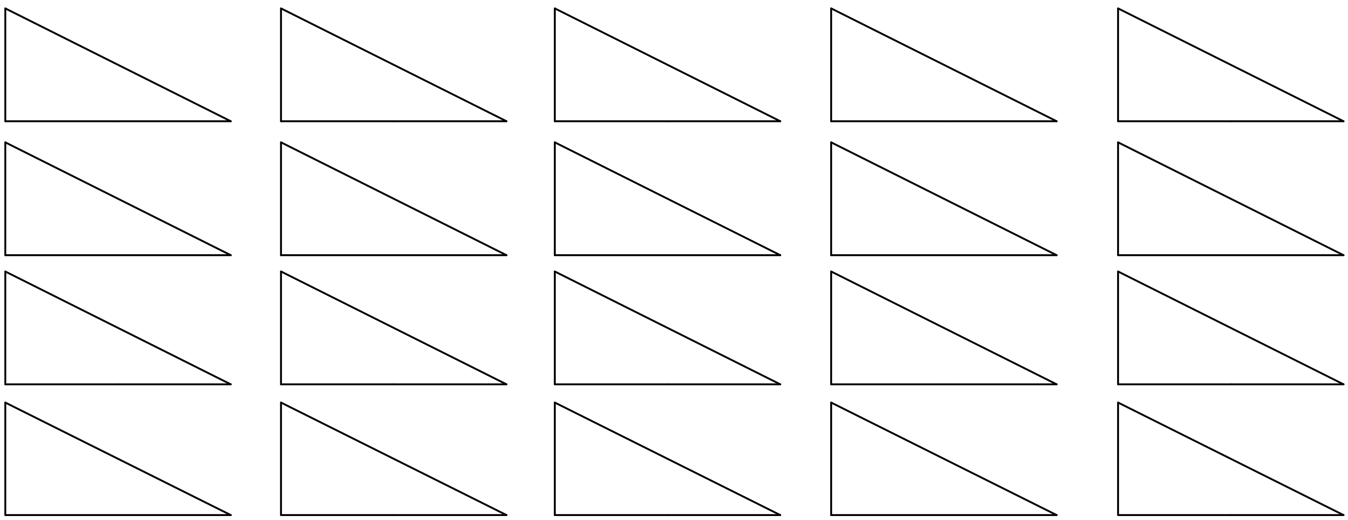
Select **RUN-MAT** and **GRAPH** mode from the **Main Menu** by using the arrow keys to highlight the **RUN-MAT** or **GRAPH** icon or by pressing the **[1]** and **[3]** keys respectively.



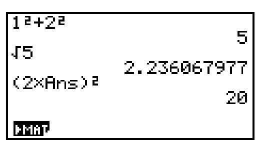


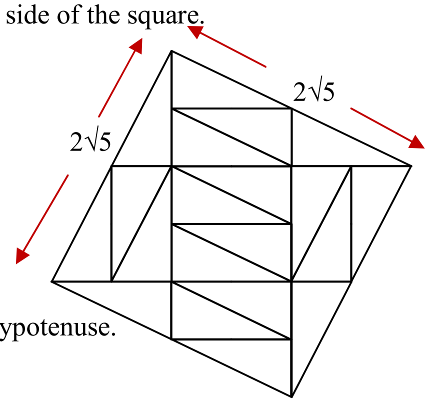
Each triangle is of an area of 1 unit, with the short sides of length 1 mu and 2 mu respectively.

Make a square from these 20 triangles below.
Cut out each of the triangles and arrange them so that a solid square is formed (i.e. there are NO gaps, no overlapping.)



Area of the square to be made = 20 mu.
Let the side of the square be of length x.
$x^2 = 20$
Applying the Pythagorean Theorem: $1^2 + 2^2 = 1 + 4 = 5$ so the hypotenuse of each triangle of $\sqrt{5}$ mu.
$\sqrt{x^2} = \sqrt{20} = \sqrt{(4\times5)} = \sqrt{4}\times\sqrt{5} = 2\sqrt{5}$, so two hypotenuses will form the length of each side of the square.





If we were to take the lengths of the short sides to be whole numbers then what other right-angled triangles can form a square in this way?
Let the side lengths be 'a' and 'a + 1' respectively: $a^2 + (a + 1)^2$ is the length of the hypotenuse.
So $2\times\sqrt{(a^2 + (a + 1)^2)} = 2\sqrt{(2a^2 + 2a + 1)}$ is the length of each side of the square.
This give the area of the square to be: $4(2a^2 + 2a + 1) = 8a^2 + 8a + 4$.

Making a table:

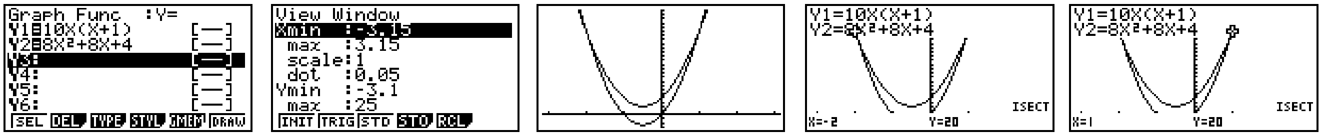
Short side length	Short side length	Hypotenuse length	Area of each triangle	Area of 20 triangles	Area of the square
a	a + 1	$2\sqrt{(2a^2 + 2a + 1)}$	½a(a+1)	10a(a+1)	$8a^2 + 8a + 4$
1	2	$2\sqrt{5}$	1	20	20 = 4×5
2	3	$2\sqrt{13}$	3	60	52 = 4×13
3	4	$2\sqrt{25}$	6	120	100 = 4×25
4	5	$2\sqrt{41}$	10	200	164 = 4×41
5	6	$2\sqrt{61}$	15	300	244 = 4×61
…	…	…	…	…	…

As you can see from the table the only way that 20 triangles can be formed with a difference of 1 appears to only be when a = 1.

Let's apply some algebra!

Area of 20 triangles = $10a(a+1)$ Area of the square = $8a^2 + 8a + 4$

For these two area s to be the same: $10a(a+1) = 8a^2 + 8a + 4$.



Enter the equations **Set the V-window** **Draw [F6]** **G-Solv** →

Or expanding gives: $10a^2 + 10a = 8a^2 + 8a + 4$

$2a^2 + 2a - 4 = 0$

$2(a^2 + a - 2) = 0$

$2(a+2)(a-1) = 0$

i.e. a = -2 or 1 only yielding a = 1 is the only real solution

Will this work for other right-angled triangles of different dimensions? What if the difference between the two short side lengths are:

2, i.e. a and a + 2?

Short side length	Short side length	Hypotenuse length	Area of each triangle	Area of 20 triangles	Area of the square
a	a + 2	$2\sqrt{(2a^2 + 2a + 1)}$	½a(a+1)	10a(a+1)	$8a^2 + 8a + 4$
1	2	$2\sqrt{5}$	1	20	20 = 4×5
2	3	$2\sqrt{13}$	3	60	52 = 4×13
3	4	$2\sqrt{25}$	6	120	100 = 4×25
4	5	$2\sqrt{41}$	10	200	164 = 4×41
5	6	$2\sqrt{61}$	15	300	244 = 4×61
…	…	…	…	…	…

3, i.e. a and a + 3?

Short side length	Short side length	Hypotenuse length	Area of each triangle	Area of 20 triangles	Area of the square
a	a + 3				

4, i.e. a and a + 4?

Short side length	Short side length	Hypotenuse length	Area of each triangle	Area of 20 triangles	Area of the square
a	a + 4				

n, i.e. a and a + n?

Short side length	Short side length	Hypotenuse length	Area of each triangle	Area of 20 triangles	Area of the square
a	a + n				

Investigate!