Compound and double angle formulas.

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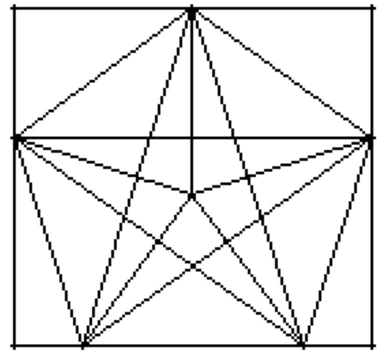
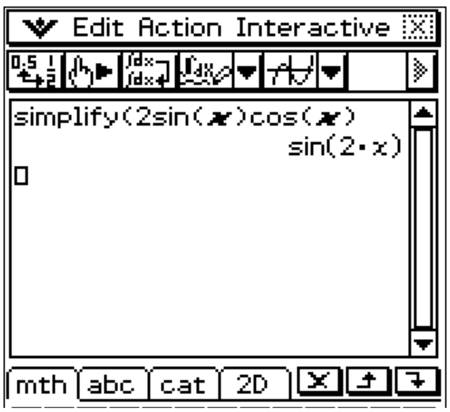
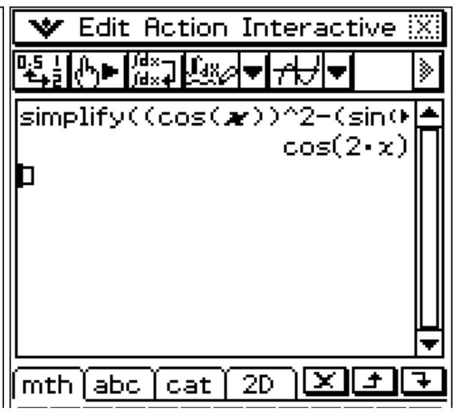
What is the relationship between the following angles, 18º, 36 º, 54º, 72º and 90º $[\frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}, \frac{2\pi}{5}, \frac{\pi}{2}]$? Can you identify each of these angles in this pentogram?



Question:
Use the compound and double angle formula to write the following in terms of A only:

a. sin(3A) using sin(A) only
b. cos(3A) using cos(A) only.





Answer:

a. $\sin(3A) = \sin(2A + A) = \sin(2A)\cos(A) + \cos(2A)\sin(A)$
$= [\sin(A+A)]\cos(A) + [\cos(A+A)]\sin(A)$
$= [2\sin(A)\cos(A)]\cos(A) + [\cos^2(A) - \sin^2(A)]\sin(A)$
$= 2\sin(A)\cos^2(A) + [\cos^2(A)\sin(A) - \sin^3(A)]$
$= 3\sin(A)\cos^2(A) - \sin^3(A)$
$= 3\sin(A)[1 - \sin^2(A)] - \sin^3(A)$
$= 3\sin(A) - 4\sin^3(A)$

b. $\cos(3A) = \cos(2A + A) = \cos(2A)\cos(A) - \sin(2A)\sin(A)$
$= [\cos(A+A)]\cos(A) - [\sin(A+A)]\sin(A)$
$= [\cos^2(A) - \sin^2(A)]\cos(A) - [2\sin(A)\cos(A)]\sin(A)$
$= \cos^3(A) - \sin^2(A)\cos(A) - 2\sin^2(A)\cos(A)$
$= \cos^3(A) - 3\sin^2(A)\cos(A)$
$= \cos^3(A) - 3[1 - \cos^2(A)]\cos(A)$
$= 4\cos^3(A) - 3\cos(A)$

Extension: What is tan(3A) using tan(A) only?

How can you calculate exactly cos(π/5) × cos(2π/5)?
Let $A = \cos(\pi/5) \times \cos(2\pi/5)$
Since $\sin(2x) = 2\sin(x) \times \cos(x)$ then $\cos(x) = ½\sin(2x)/\sin(x) \rightarrow (1\ )$
When $x = \pi/5$ then $\cos(\pi/5) = ½\sin(2\pi/5)/\sin(\pi/5)$

From Equation (1) $[\sin(2\pi/5) \times \cos(2\pi/5)] = ½\sin(4\pi/5)$
since $A = \cos(\pi/5) \times \cos(2\pi/5)$

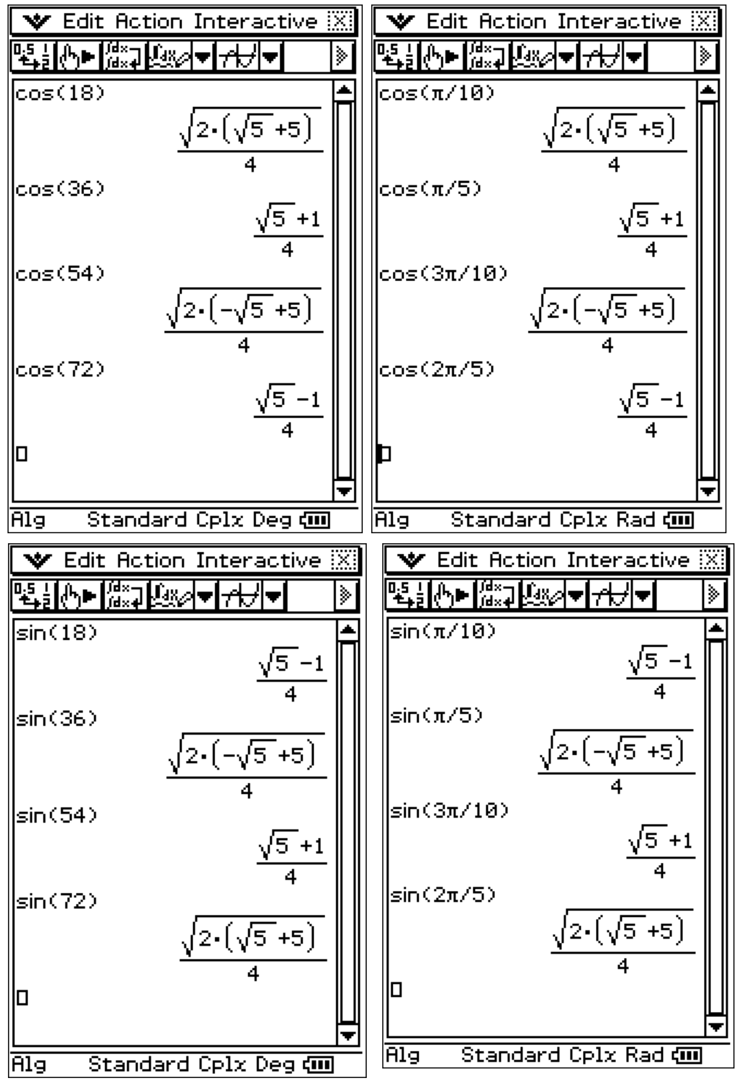
$= ½\sin(2\pi/5) \times \cos(2\pi/5)] / \sin(\pi/5)$
$= ½[\sin(4\times\pi/5] / [2 \times \sin(\pi/5)]$

But $\sin(4\pi/5) = \sin(\pi - \pi/5)$
$= \sin(\pi)\times \cos(\pi/5) - \cos(\pi)\times \sin(\pi/5)$
$= \sin(\pi/5)$

Since $A = ½[\sin(4\times\pi/5)] / [2 \times \sin(\pi/5)]$
$= ½[\sin(\pi/5)]/[2 \times \sin(\pi/5)]$
$= ¼$

As $\pi/5 + 4\pi/5 = \pi$ it follows that $\sin \pi/5 = \sin(4\pi/5)$

Find the exact values of cos(2π/5) and cos(4π/5)?



$\cos(5x) = 16\cos^5(x) - 20\cos^3(x) + 5\cos(x)$

Solving this quintic $16x^5 - 20x^3 + 5x - 1 = 0$

There is no general solution for a 5th order polynomial.

However, the half and double angle formulae can be used to solve either problem.

Solving for $\cos(\pi/5)$ will give you the desired answer.

Note: $\sin(3\times\pi/5) = \sin(2\times\pi/5)$

The triple angle sine formula is:

$\sin(3x) = 4\sin(x) \times \cos^2(x) - \sin(x)$

The double angle sine formula is: $\sin(2x) = 2 \sin(x) \times \cos(x)$

Since $\sin(3x) = \sin(2x)$ when $x = \pi/5$

$4\sin(x) \times \cos^2(x) - \sin(x) = 2 \text{ in}(x) \times \cos(x)$

Divide out the sin(x) term gives $4\cos^2 (x) - 1 = \cos(x)$

Giving a simple quadratic in $\cos(x)$,

substituting $y = \cos(\pi/5)$ gives:

$4y^2 - 2y - 1 = 0$ and solving for y gives 2 solutions:

$y = (1 + \sqrt{5})/4$ and $y = (1 - \sqrt{5})/4$.

Since $\cos(\pi/5)$ is positive then $\cos(\pi/5) = (1 + \sqrt{5})/4$

Use the double angle formula:

$\cos(2x) = 2\cos^2(x) - 1$

$\cos(2\pi/5) = 2\times(1 + \sqrt{5})/4)^2 - 1 = (\sqrt{5} - 1)/4$

Note: $\cos(4\pi/5) = - \cos(\pi/5)$ so $\cos(4\pi/5) = - (1 + \sqrt{5})/4$

If sin(3π/10) = (1 + √5)/4, find the exact value of cos(π/5)

Note that $½\pi = 3\pi/10 = \pi/5$ and that $\cos(½\pi - x) = \sin(x)$ using $x = 3\pi/10$

$\cos(\pi/5) = \cos[(½\pi) - (3\pi/10)]$

$\cos(\pi/5) = \cos(½\pi) \times \cos(3\pi/10) + \sin(\pi/2) \times \sin(3\pi/10)$

$\cos(\pi/5) = \sin(3\pi/10)$

$\cos(\pi/5) = (1 + \sqrt{5})/4$